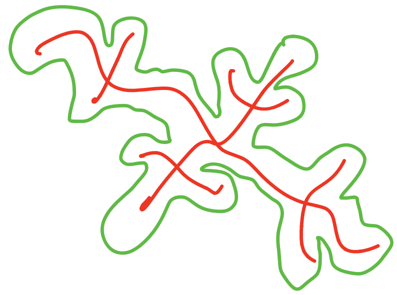
Holomorphic Dynamics - Lecture 14

Def: A rational map is <u>hyperbolic</u> if one of the two equivalent properties hold:

1. There is a metric ρ on a nbd of $J(f)$ s.t.
   $$\|Df\|_\rho \geq c > 1$$



2. Every critical point converges to an attracting cycle

Example $f_c(z) = z^2 + c, \qquad c \in \mathbb{C}$

Rmk hyperbolicity is an <u>open</u> condition in parameter space

Cor.: A quadratic polynomial is hyperbolic iff the (finite) critical point converges to an attracting cycle.

Case 1 For what c there is an attracting fixed pt?

$$\begin{cases} f_c(z_0) = z_0 \\ |f_c'(z_0)| < 1 \end{cases} \longleftrightarrow \qquad 2|z_0| < 1$$

$$|z_0| < \frac{1}{2} \qquad z_0 = \frac{1}{2}\rho\, e^{i\theta} \qquad \rho < 1$$

$$z_0 = z_0^2 + c$$

$$c = z_0 - z_0^2 = \frac{1}{2}\rho\, e^{i\theta} - \frac{1}{4}\rho^2 e^{2i\theta}$$

E.g.: $c = 0$ belongs to hyp. component.

If $\rho = 1$ you find boundary of component.

$$\partial H = \left\{ \frac{1}{2} e^{i\theta} - \frac{1}{4} e^{2i\theta} \ , \ 0 \leq \theta \leq 2\pi \right\}$$

main cardioid

Def.: The MANDELBROT SET is

$$\mathcal{M} := \{ c \in \mathbb{C} : \ (f_c^n(0)) \text{ is bounded} \}$$

and

$$\mathbb{C} \setminus \mathcal{M} = \{ c \in \mathbb{C} : \ \lim_{n\to\infty} f_c^n(0) = \infty \}$$

Def.: A connected component of the set of hyperbolic quadratic polynomials is called a HYPERBOLIC COMPONENT

Cor.: $\mathbb{C} \setminus \mathcal{M}$ is a hyperbolic component

For every $c \in \mathbb{C} \setminus \mathcal{M}$, $f_c^n(0) \longrightarrow \infty$

and $J(f_c)$ is totally disconnected,

(hence a Cantor set).

Hyperbolic Component(s?) of Period 2

$$f_c(f_c(z_0)) = z_0$$

$$\lambda = f_c'(f_c(z_0))\, f_c'(z_0) \quad , \quad |\lambda| < 1$$

$$(z^2+c)^2 + c = z$$

$$\begin{cases} z^4 + 2cz^2 + c^2 + c = z \\ \lambda = 2(z^2+c)\,2z = \rho e^{i\theta} \end{cases}$$

To find bdry, set $\rho = 1$

$$\begin{cases} (z^2+c)^2 + c = z \\ z(z^2+c) = \frac{1}{4} e^{i\theta} \end{cases}$$

$$z^2 + c = \frac{e^{i\theta}}{4z}$$

$$\left(\frac{e^{i\theta}}{4z}\right)^2 + c = z$$

$$c = z - \frac{e^{2i\theta}}{16 z^2} \rightarrow \text{Equation for component ?}$$

<u>Center of component</u> : superattracting critical point

$$0 \rightarrow c \rightarrow c^2 + c = 0$$

$c=0 \rightarrow$ center of cardioid

$c=-1 \rightarrow$ center of period 2 component

Def.: Each hyperbolic component (for quadratic polynomials) has a UNIQUE parameter c for which 0 is superattracting critical point. Such parameter is called the <u>CENTER</u>

Period 3

$$0 \rightarrow c \rightarrow c^2 + c \rightarrow (c^2+c)^2 + c = 0$$

$$c^4 + 2c^3 + c^2 + c = 0$$

$c=0 \rightarrow$ cardioid (period 1)

$$c^3 + 2c^2 + c + 1 = 0$$

$c \approx -1.75488$ AIRPLANE

$c \approx -0.122561 \pm 0.744862i$

For every period p, there are finitely many components of such period.

CONJECTURE (DENSITY OF HYPERBOLICITY)

Every $c \in \mathbb{C}$ lies in the closure of the union of all hyperbolic components,

TRY YOURSELF: Given $f_c(z) = z^2 + c$, for any $\varepsilon > 0$ find c' with $|c' - c| < \varepsilon$ s.t. $f_{c'}^n(0)$ converges to a periodic attracting cycle.